CALCULATION OF INTERFACIAL FORCES FOR AN IDEAL FLUID WITH A RANDOM DISTRIBUTION OF BUBBLES

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In order to compute the the motion of bubbles relative to a fluid, it is necessary to know the forces acting on them. In a dispersed medium the interaction between inclusions via the carrier phase can significantly affect the forces acting on the inclusions, and this force depends on the microstructure of the medium [1]. Two limiting cases are a regular distribution, where the distance between neighboring inclusions is approximately constant, and a random distribution, where the inclusions are distributed randomly.

Calculation of the interfacial force in a medium with bubbles in the special case where there are no gradients of the flow parameters reduces (as shown in [2], for example) to a calculation of the effective mass of a bubble in the mixture. The effective mass of inclusions in a dispersed medium is calculated in many papers under various assumptions. An analysis and review of these results is given in [2].

In the general case where there are gradients of the flow parameters, the interfacial force for a fluid with bubbles has been studied much less. A detailed analysis in this case has been considered only for mixtures with a regular structure. In [1] the cell method is used to study the interfacial force and a review of the literature is given.

In the present paper we present a method based on the averaging of the microequations of the interfacial interactions for the case of mixtures with a random structure. Unlike [2], we consider the case where the characteristics of the dispersed medium vary along the z axis. The effect of gradients of the average parameters on the interfacial force is studied.

1. Basic Assumptions

We assume that the carrier medium is an ideal fluid and the bubbles are spherical. It was shown in [3] that these assumptions are valid for bubbles of moderate size. In addition, we assume that the medium is monodispersed, that the flow of the carrier phase is potential flow, and that the distribution of bubble centers is random [2, 4]. For simplicity we consider the one-dimensional form of the problem: the average parameters of the medium vary only along the z axis and do not depend on the other coordinates (x, y).

The average force acting on an isolated (test) bubble is determined by averaging the force acting on the bubbles when their distribution relative to each other is given. In order to calculate the force acting on a test bubble with a given distribution of the other bubbles, it is necessary to know the translational velocities of the bubbles, which in turn depend on the forces. In general, the problem cannot be solved. Therefore, in the present paper we study an approximation method for determining the velocities of the bubbles and then calculate the average force. It was shown in [2] that it is physically justifiable to assume that the velocities of the bubbles relative to the fluid are constants for any distribution of bubbles. This assumption qualitatively reflects the basic features of the motion of bubbles in a carrier medium; in particular, it describes the effect of a more rapid translation of two bubbles moving one behind the other in comparison with the translation of a single bubble and also the displacement of the velocity vector of a pair of bubbles in the direction of the line joining their centers (in comparison with the velocity vector of a single bubble).

Following [2], the z component of the velocity of the i-th bubble Δv relative to the fluid for a given distribution of other inclusions is calculated from the equation $\Delta v = -2k/R^3$, where k is the coefficient of the associated Legendre function of the first kind $P_1^0(\cos\theta)$ in the expansion of the velocity potential of the carrier medium near the i-th bubble. It is easy to show that this relation reduces to the usual one for the motion of a

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single bubble in an infinite fluid. The components of the velocity in the x and y directions are determined in a similar way.

2. Basic Equations

The average interfacial force is calculated as follows. The velocity profile of the fluid for a given distribution of bubbles is calculated with the necessary degree of accuracy. Using the known velocity profile, the pressure distribution on the surface of a fixed bubble is determined, and thus the force acting on it is calculated. Averaging the force acting on the bubble with respect to the positions of the bubbles gives the average interfacial force.

The velocity potential φ of the fluid in the region near the bubble satisfies the equation

$$\Delta\varphi = 0, \tag{2.1}$$

and in order to solve this equation it is necessary to know the boundary conditions. On the boundary Γ_0 of the region G occupied by the mixture, it is natural to specify the normal component of the velocity v^0

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\Gamma_0} = \mathbf{n}\mathbf{v}^0(\Gamma_0) \tag{2.2}$$

(n is the normal to the surface).

On the surfaces of the bubbles Γ_i (i is the number of the bubble, i = 1, 2, ..., N, where N is the number of bubbles in the mixture) the condition of nonpenetrability of fluid can be written as

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\Gamma_i} = \mathbf{n}\mathbf{v}_2^i \tag{2.3}$$

($v_2{}^i$ is the velocity of motion of the i-th bubble). The vectors $v_2{}^i$ are chosen such that the velocity of a bubble relative to the fluid is equal to $\Delta v(r)$, which is a given smooth function of position.

Equation (2.1), with the boundary conditions (2.2) and (2.3), determines the velocity potential of the fluid, to within an insignificant constant; its solution will be constructed by successive approximations, giving the perturbations on groups of bubbles with respect to the others and the boundaries of the region [2, 5].

The first approximation φ^1 is written in the form

$$\varphi^1 = \sum_{i=1}^{N} \varphi_i^1, \tag{2.4}$$

where φ_i^1 is the velocity potential of the fluid from the motion of the i-th bubble in an infinite fluid.

The second approximation φ^2 is found from the condition that the sum $\varphi^1 + \varphi^2$ satisfies (2.1) and the conditions on the boundary Γ_0 of region G occupied by the mixture. In view of the linearity of the problem and the representation (2.4), the potential

$$\varphi^2 = \varphi_0^2 + \sum_{i=1}^{N} \varphi_i^2.$$

Here φ_0^2 does not depend on the positions of the bubbles, while φ_i^2 depends only on the position of the i-th bubble.

The third approximation φ^3 is written in a form similar to the first:

$$\varphi^3 = \sum_{i=1}^{N} \varphi_i^3.$$

The potential $\varphi = \varphi^1 + \varphi^2 + \varphi^3$ satisfies the boundary conditions on the surface of the i-th bubble.

The successive approximations are constructed in a similar way. The equations satisfied by φ_i^1, φ^2 are given in [2].

The exact solution of (2.1) with the boundary conditions (2.2) and (2.3) is given by

$$\varphi = \sum_{k=1}^{\infty} \varphi^k. \tag{2.5}$$

The velocity potential of the fluid near two moving bubbles constructed in this way, as shown in [5], rapidly converges to the exact potential for any distance between the inclusions (including the case where they are touching).

It is easy to show that among the terms of (2.5) there is one which does not depend on the positions of the bubbles (φ_0^2), two which depend on the position of one bubble (φ_i^1, φ_i^2), terms which depend on the positions of two bubbles (there are such terms in φ^3), etc. Therefore, (2.5) for the potential is conveniently rewritten as

$$\varphi = \varphi(\mathbf{r} \mid \mathbf{r}_{1i} \ldots, \mathbf{r}_N) = \sum_{l=0}^{N} \chi_l^0(\mathbf{r}); \tag{2.6}$$

$$\chi_l^0 = \sum_{w_{i1,\ldots,il}^N} \chi_l(\mathbf{r} \mid \mathbf{r}_{i1}, \ldots, \mathbf{r}_{il}). \tag{2.7}$$

The summation in (2.7) goes over all combinations $w_{i1}{}^N{}_{,\ldots,i\ell}$ of ℓ bubbles taken out of N. The function $\chi_\ell(\mathbf{r} \mid \mathbf{r}_{i1}, \ldots, \mathbf{r}_{i\ell})$ depends on the positions of ℓ bubbles with numbers il, ..., iℓ. The quantity $\chi_\ell{}^0$ will be called the ℓ-th partial term of the potential.

The force acting on a bubble in an ideal inviscid carrier medium is determined by the pressure distribution around the bubble and can be calculated from the formula

$$F = -\int_{\Gamma_1} p\mathbf{n}ds, \tag{2.8}$$

where Γ_1 is the surface of the test bubble, and p is the pressure in the fluid.

The pressure is found from the Cauchy–Lagrange integral [6]

$$p = \rho f(t) + \rho U - \rho \frac{\partial \varphi}{\partial t} - \rho \frac{(\nabla \varphi)^2}{2}. \tag{2.9}$$

Here φ is the velocity potential of the fluid; t is the time; f(t) is a function of time; U is the potential of the body force, here taken to be the force of gravity.

We substitute (2.9) into (2.8). The first term of (2.9) gives a zero contribution, and the second term gives the buoyance force $F_A = {}^4/_3 \pi R^3 \rho g \mathbf{e}_z$ (R is the radius of the bubble, g is the acceleration of gravity, and $\mathbf{e}_z$ is a unit vector along the z axis). The other terms in the force F depend upon the positions of all of the bubbles and, therefore, we isolate one of the bubbles (let it have the number 1) and average the force over the positions of all the other bubbles, i.e., we multiply by the distribution function f_N and integrate over the state space. The final expression for the average force is:

$$\langle F \rangle = F_A + F_1 + F_2 = F_A + \rho \left\langle \int_{\Gamma_1} \frac{\partial \varphi}{\partial t} \cos\theta ds \right\rangle + \frac{\rho}{2} \left\langle \int_{\Gamma_1} (\nabla \varphi)^2 \cos\theta ds \right\rangle, \tag{2.10}$$

where < > denotes an average over the positions of the bubbles; θ is the angle between the radius vector at the point where the potential is being computed and the z axis.

Formulas (2.6) and (2.7) can then be written in the form:

$$\varphi = \varphi(\mathbf{r}, \mathbf{r}_1 | \mathbf{r}_2, \ldots, \mathbf{r}_N) = \sum_{l=0}^{N-1} \chi_{lf}^0(\mathbf{r}); \tag{2.11}$$

$$\chi_{lf}^0 = \sum_{w_{i1,\ldots,il}^N} \chi_{lf}(\mathbf{r}, \mathbf{r}_1 | \mathbf{r}_{i1}, \ldots, \mathbf{r}_{il}). \tag{2.12}$$

Here the function $\chi_{\ell f}(\mathbf{r}, \mathbf{r}_1|\mathbf{r}_{i1}, \ldots, \mathbf{r}_{i\ell})$ contains all terms of the total potential dependent upon the positions of the bubbles i1, ..., iℓ. It is not important whether it depends on the position of the first (test) bubble or not.

3. Calculation of the Term F_1 in (2.10)

Using the fact that $N \gg 1$, the term F_1 can be written as

$$F_1 = \rho \left\langle \int_{\Gamma_1} \frac{\partial \varphi}{\partial t} \cos\theta ds \right\rangle = \rho \sum_{l=0}^{N} (\alpha_2)^l \frac{1}{l}\left(\frac{3}{4\pi R^3}\right)^l \times$$
$$\times \int_{r_2} \ldots \int_{r_{l+1}} \int_{\Gamma_1} \frac{\partial}{\partial t} \chi_{lf}(\mathbf{r}, \mathbf{r}_1 | \mathbf{r}_2, \ldots, \mathbf{r}_{l+1}) \times$$
$$\times f_l(\mathbf{r}_1 | \mathbf{r}_2, \ldots, \mathbf{r}_{l+1}) \cos\theta ds d^3r_2 \ldots d^3r_{l+1}, \tag{3.1}$$

where α_2 is the volume concentration of bubbles and $f_f(\mathbf{r}_1|\mathbf{r}_2, \ldots, \mathbf{r}_{\ell+1})$ is the ℓ-th partial distribution function of the centers of the bubbles [7] subject to the condition that the first bubble is found at point $\mathbf{r}_1$. The representation (3.1) correctly gives the expansion of the original integral in a power series in α_2 in the case when the integrals on the right-hand side of (3.1) have finite limits when $G \to \infty$ and $\alpha_2 \to 0$ (G is the volume of the mixture).

For a medium with a regular structure, these integrals diverge [2]. However, for a medium with a random structure whose average characteristics do not vary along the z axis, $\partial\chi_{\ell j}/\partial t$ was calculated in [2] and it was shown there that the integrals on the right-hand side of (3.1) are convergent and finite for $\ell \geq 2$, and this is also true in the case considered here. We will calculate the force to first order in α_2. It is then sufficient to calculate the integrals of $\partial\chi_{0f}/\partial t$ and $\partial\chi_{1f}/\partial t$. In order to do this, it is necessary to know all of the approximations in the iteration scheme, but only taking into account two bubbles. This means, in effect, that the exact solution of the two-bubble problem must be found. It can be shown [2] that the contributions from the third and higher approximations can be found directly from (3.1) by numerical methods. The contribution from the first two approximations is considerably more complicated to calculate. This is because, beginning with the third approximation, the potential falls off more rapidly than a quadrupole potential, and φ^1 contains a dipole component. Therefore, it is convenient to consider the contribution from $\varphi = \varphi^1 + \varphi^2$ separately. The averaging procedure is simpler and there is no loss of generality in assuming that $\mathbf{r}_1 = 0$, although $v_2{}^1 \neq 0$. Let the coordinate system be referred to the boundary Γ_0 of the region occupied by the mixture. We introduce the notation:

$$A_0^i(\mathbf{r}) = \frac{1}{|\mathbf{r} - \mathbf{r}_i|}, \quad A_1^i = \frac{\partial A_0^i}{\partial z}, \quad A_2^i = \frac{\partial^2 A_0^i}{(\partial z)^2}, \quad A_3^i = \frac{\partial^3 A_0^i}{(\partial z)^3}, \quad A_{2x}^i = \frac{\partial^2 A_0^i}{\partial z \partial x}, \quad A_{2y}^i = \frac{\partial^2 A_0^i}{\partial z \partial y}.$$

Then from the algorithm for constructing φ we obtain

$$\varphi^1 + \varphi^2 = \sum_{i=1}^{N} \frac{R^3 \Delta v(\mathbf{r}_i, t)}{2} A_1^i + \varphi^2; \tag{3.2}$$

$$\frac{\partial(\varphi^1 + \varphi^2)}{\partial t} = \frac{R^3}{2} \sum_{i=1}^{N} \left\{ A_1^i \left(\left.\frac{\partial \Delta v(\mathbf{r}, t)}{\partial t}\right|_{\mathbf{r}=\mathbf{r}_i} + \left.\frac{\partial \Delta v(\mathbf{r}, t)}{\partial z}\right|_{\mathbf{r}=\mathbf{r}_i} v_{2z}^i(\mathbf{r}_i, t) \right) - \right.$$
$$\left. - \Delta v(\mathbf{r}_i, t)\left[A_2^i v_{2z}^i + A_{2x}^i v_{2x}^i + A_{2y}^i v_{2y}^i\right] \right\} + \frac{\partial \varphi^2}{\partial t} \tag{3.3}$$

[$v_2{}^i(v_{2x}{}^i, v_{2y}{}^i, v_{2z}{}^i)$ is the velocity of the i-th bubble].

It is impossible to average (3.3) directly over the positions of the bubbles because φ^1 does not fall off rapidly enough and the corresponding integral will diverge. Therefore, in averaging (3.3) we use the method of [2]. The quantity $\partial(\varphi^1 + \varphi^2)/\partial t$ can be related to the average parameters of the fluid. Then the difference can be expressed in terms of convergent integrals, and in the final expression we put the average parameters of the fluid. In order to do this, we make use of Green's relation

$$\int\limits_V (\psi\Delta\Phi - \Phi\Delta\psi)\, d^3V = \int\limits_S \left(\psi\frac{\partial\Phi}{\partial n} - \Phi\frac{\partial\psi}{\partial n}\right) d^2s, \tag{3.4}$$

where V is an arbitrary volume; S is its surface; ψ and Φ are arbitrary twice-differentiable functions.

As the volume V we take G minus the volume of the test bubble, and for ψ and Φ we use $\psi_1 = \partial(\varphi^1 + \varphi^2)/\partial t$ and $\Phi_1 = \cos\theta/|\mathbf{r}|^2$, respectively. Then after some straightforward reductions, taking into account that $\Delta A_0^{\,i} = -4\pi\delta(\mathbf{r} - \mathbf{r}_i)$ and $\Delta A_1^{\,i} = -4\pi\partial[\delta(\mathbf{r} - \mathbf{r}_i)]/\partial z$ [9], we have

$$\int\limits_{\Gamma_0} \left[\psi_1 \frac{\partial\Phi_1}{\partial n} - \Phi_1 \frac{\partial\psi_1}{\partial n}\right] d^2s = \int\limits_{\Gamma_1} \psi_1 2\frac{\cos\theta}{R^3}\, d^2s + \int\limits_{\Gamma_1} \frac{\cos\theta}{R^2}\frac{\partial\psi_1}{\partial n}\, d^2s + p_1; \tag{3.5}$$

$$p_1 = 4\pi\frac{R^3}{2}\sum_{i=2}^{N} p_{1i}, \tag{3.6}$$

$$p_{1i} = A_2^i(0)\, k_1^i + \Delta v(\mathbf{r}_i, t)\left[v_{2z}^i A_3^i(0) + v_{2x}^i A_{3x}^i(0) + v_{2y}^i A_{3y}^i(0)\right],$$

$$k_1^i = \left.\frac{\partial\Delta v(\mathbf{r}, t)}{\partial t}\right|_{\mathbf{r}=\mathbf{r}_i} + \left.\frac{\partial\Delta v(\mathbf{r}, t)}{\partial z}\right|_{\mathbf{r}=\mathbf{r}_i} v_{2z}^i.$$

The first term on the right-hand side of (3.5) is proportional to the contribution $\varphi^1 + \varphi^2$ in the force F due to the term $\partial\varphi/\partial t$, i.e., it is proportional to the desired quantity.

However, several quantities which are in general unknown appear in (3.5) to a more convenient form. This can be done applying (3.4) two more times: the first time, $V = G_1$ (the volume of the test bubble), and for ψ we use $\psi_1 = \partial(\varphi^1 + \varphi^2)/\partial t$, while for Φ we use $\Phi_2 = |\mathbf{r}|\cos\theta$; the second time the distribution of bubbles is such that the observation point (origin of the coordinate system) is in the fluid. The volume is $V = G$. For ψ we use $\psi_2 = \partial(\varphi^1 + \varphi^2)/\partial t$, and for Φ we use $\Phi_1 = \cos\theta/|\mathbf{r}|^2$. Completing the calculations and averaging over the positions of the bubbles, (3.5) takes the form

$$\frac{3}{R^3}\left\langle \int\limits_{\Gamma_1} \psi_1 \cos\theta\, d^2s \right\rangle = \int\limits_{\Gamma_0} \frac{\partial\Phi_1}{\partial n}\left(\langle\psi_2\rangle_l - \langle\psi_1\rangle\right) d^2s + 4\pi\frac{R^3}{2}\int\limits_G n_1\left[p_{12}\, g\,(|\mathbf{r}_2|) - \right. \tag{3.7}$$

$$\left. - (p_{12})_l\, g_l(|\mathbf{r}_2|)\right] d^3r_2 + 4\pi\left\langle \frac{\partial}{\partial z}\psi_2 \right\rangle - \frac{4\pi}{R^3}\frac{R^3}{2}k_1^1,$$

where the subscript ℓ corresponds to the case where the observation point is in the fluid; n_1 is the number of bubbles per unit volume of the mixture; $g(|\mathbf{r}_2|)$ is the binary correlation function [7]; $g_\ell(|\mathbf{r}_2|)$ is the ratio of the probability of finding a bubble at a distance $|\mathbf{r}_2|$ from the observation point (inside the fluid) to n_1.

The quantities p_{12} and $(p_{12})_\ell$ are given by the same formula (3.6), but differ because they involve the velocities of the bubbles, which depend on whether the bubble is at the observation point or not. In general, the velocity $v_2^{\,i}$ depends on the positions of all of the bubbles. However, the integral over region G on the right-hand side of (3.7) is already of order α_2. Therefore, the integrand and the velocity of the bubble can be calculated to order $(\alpha_2)^0$. It can be shown that, in this case, in the calculation of $v_2^{\,i}$ entering into $(p_{12})_\ell$, it is necessary to take into account only the velocity of the carrier medium and the velocity of slipping Δv of the second bubble relative to the carrier medium. In the calculation of $v_2^{\,i}$ entering into p_{12}, it is necessary to take into account the velocity of the carrier medium, the velocity of slipping of the second bubble relative to the carrier medium, and the correction to the velocity of the second bubble induced by the first bubble. The contribution of the other bubbles to the velocity of motion of the second bubble is of order α_2 when averaged over the positions of the other bubbles and can be neglected.

Unfortunately, even this simplification of the problem does not permit an analytical calculation of the velocity of the second bubble entering into p_{12}, because, in order to do this, it would be necessary to find an exact solution of the problem of the motion of two bubbles in an infinite fluid. The solution of this problem, as found by the method of successive approximations, is in the form of an infinite series which, as already noted above, rapidly converges. Therefore, in the present paper, the velocity of the second bubble is calculated by keeping only the first term of this series.

The quantities $g(|r_2|)$ and $g_\ell(|r_2|)$ in (3.7) are calculated to order $(\alpha_2)^0$, which means that for a random distribution of bubbles they are given by the formulas [4]

$$g(|\mathbf{r}_2|)=\begin{cases}0 \text{ when } |\mathbf{r}_2|\leqslant 2R,\\ 1 \text{ when } |\mathbf{r}_2|>2R,\end{cases}\qquad g_l(|\mathbf{r}_2|)=\begin{cases}0 \text{ when } |\mathbf{r}_2|\leqslant R,\\ 1 \text{ when } |\mathbf{r}_2|>R.\end{cases}$$

Finally, it can easily be shown that the first term on the right-hand side of (3.7) is of order $1/L$ (L is the characteristic linear dimension of the region G) and can be neglected.

Using the above simplifications, after lengthy calculation it can be shown that (3.7) becomes

$$\frac{3}{R^3}\left\langle\int_{\Gamma_1}\frac{\partial}{\partial t}(\varphi^1+\varphi^2)\cos\theta d^2s\right\rangle=4\pi\left\{\frac{1}{40}\,\alpha_2\Delta v\frac{\partial}{\partial z}\Delta v-\right.$$
$$\left.-\frac{3}{40}\Delta v\frac{\partial}{\partial t}(\alpha_2\Delta v)+\left\langle\frac{\partial\psi_2}{\partial z}\right\rangle_l-\frac{1}{2}\left(\frac{\partial\Delta v(\mathbf{r},t)}{\partial t}+\frac{\partial\Delta v(\mathbf{r},t)}{\partial z}\right)\langle v^1_{2z}\rangle\right\}. \tag{3.8}$$

Since we took into account the contribution of $\varphi^1+\varphi^2$, in F_1 of (2.10) separately, the contribution F_{1d} of the remaining terms of the series for the exact potential φ, can be taken into account directly with the use of the expansion (3.1):

$$F_{1d}=\alpha_2\frac{3}{4\pi R^3}\int_{\Gamma_1}\int_{r_2}\frac{\partial}{\partial t}\chi_1(\mathbf{r},\mathbf{r}_1|\mathbf{r}_2)f_1(\mathbf{r}_1|\mathbf{r}_2)\cos\theta d^2sd^3r_2.$$

The potential χ_1 cannot be evaluated analytically (it can be represented as an infinite series). As noted above, we can limit the expansion of the potential φ to the first principal term after the terms $\varphi^1+\varphi^2$ which have already been taken into account. The result is that the contribution of this term to the force acting on a bubble is equal to zero. Hence, (3.8) is an accurate expression for F_1.

The quantity $\langle\partial\psi_2/\partial z\rangle = \langle\partial^2(\varphi^1+\varphi^2)/\partial z\partial t)\rangle_\ell$ appears in (3.8). This quantity is conveniently expressed in terms of the derivative of the average velocity of the fluid $\langle\partial v_{\ell z}/\partial t\rangle_\ell$. The exact expression for $\langle v_{\ell z}\rangle$ has the form

$$\langle v_{lz}\rangle_l=\int_{r_1}\dots\int_{r_N}\frac{\partial}{\partial z}(\varphi^1+\varphi^2+\dots)f_N(0|\mathbf{r}_1,\dots,\mathbf{r}_N)d^3r_1\dots d^3r_N.$$

Using (2.6) and following a procedure similar to that used in deriving (3.1), it can be shown that in the calculation of the fluid velocity to order $(\alpha_2)^1$, it is sufficient to include only the single-particle interaction in (3.10); $\chi_1{}^0$ appears in φ^1 in the first approximation, and also in all of the successive approximations because of the boundaries. However, the contribution of the single-particle interaction in the approximations higher than the first is of order $1/L$ and, therefore, can be neglected in the limit $L\to\infty$. Therefore, we have the relation

$$\left\langle\frac{\partial}{\partial z}\psi_2\right\rangle_l=\left\langle\frac{\partial}{\partial t}v_{lz}\right\rangle_l. \tag{3.9}$$

For a closed model of a two-phase medium, the force must be expressed in terms of the derivatives of the average quantities, which in general are not equal to the average values of the derivatives [1]. Therefore, a relation between the quantities $\langle\partial v_{\ell z}\rangle_\ell/\partial t$ and $\langle\partial v_{\ell z}/\partial t\rangle_\ell$ would be useful. From (3.2) we have

$$v_{lz} = \sum_{i=1}^{N} \frac{R^3 \Delta v(\mathbf{r}_i, t)}{2} A_1^i + \frac{\partial \varphi^2}{\partial t}. \tag{3.10}$$

Using (3.10) we can immediately calculate $\langle \partial v_{\ell z}/\partial t \rangle_\ell$ and $\partial \langle v_{\ell z} \rangle_\ell / \partial t$ and we finally obtain

$$\left\langle \frac{\partial v_{lz}}{\partial t} \right\rangle_l - \frac{\partial \langle v_{lz} \rangle_l}{\partial t} = -\langle v_{2z} \rangle \alpha_2 \frac{\partial \Delta v}{\partial z} + \frac{3}{5} \frac{\partial \Delta v \langle v_{2z} \rangle \alpha_2}{\partial z} + \Delta v \frac{\partial \alpha_2}{\partial t}. \tag{3.11}$$

From (3.8), (3.9), (3.11) and the equation of continuity for the bubbles, we find an expression for the term F_1 in (2.10):

$$\begin{aligned} F_1 = \frac{4}{3} \pi R^3 \rho \Big\{ & \frac{1}{40} \alpha_2 \Delta v \frac{\partial \Delta v}{\partial z} - \frac{3}{40} \Delta v \frac{\partial (\alpha_2 \Delta v)}{\partial z} + \frac{\partial \langle v_{lz} \rangle_l}{\partial t} - \\ & - \frac{2}{5} \frac{\partial (\Delta v \langle v_{2z} \rangle \alpha_2)}{\partial z} - \frac{1}{2} \left(\frac{\partial \Delta v}{\partial t} + \langle v_{2z} \rangle \frac{\partial \Delta v}{\partial z} \right) \Big\}. \end{aligned} \tag{3.12}$$

4. Calculation of the Term F_2 in (2.10)

The term F_2 of the total force $\langle F \rangle$ is determined, according to (2.10), by the expression

$$F_2 = \frac{\rho}{2} \int_{\Gamma_1} \langle (\nabla \varphi)^2 \rangle \cos \theta \, d^2 s. \tag{4.1}$$

One cannot use (4.1) directly because $\nabla\varphi$ in general diverges as the size of the region G increases (if the volume concentration of inclusions is constant). Therefore, as done above, it is necessary to separate the divergent part from (4.1) and express it in terms of the average characteristics of the carrier medium. In order to do this, we use the formula

$$\langle (\nabla \varphi)^2 \rangle = \langle \nabla \varphi \rangle^2 + \langle (\nabla \varphi')^2 \rangle, \tag{4.2}$$

where

$$\nabla \varphi' = \nabla \varphi - \langle \nabla \varphi \rangle.$$

It can be shown that the second term on the right-hand side of (4.2) to order $(\alpha_2)^1$ can be expressed in terms of a convergent integral of the one-particle terms of the potential

$$\langle (\nabla \varphi')^2 \rangle = \int_{r_2} (\nabla \chi_1 (\mathbf{r}, \mathbf{r}_1 | \mathbf{r}_2) \frac{\alpha_2 (\mathbf{r}_2)}{\frac{4}{3} \pi R^3} g(|\mathbf{r}_2|) \, d^3 r_2. \tag{4.3}$$

In the first term on the right-hand side of (4.2) it is necessary to evaluate the average gradient of the potential induced by the other bubbles. As in the calculation of F_1, contributions from the quadrupole terms (they fall off as $1/|\mathbf{r}_i - \mathbf{r}_1|^4$), and from higher-order terms converge and can be calculated directly. The dipole terms fall off as $1/|\mathbf{r}_i - \mathbf{r}_1|^3$ and, therefore, the contribution from these terms diverges as the region G increases. Divergent dipole terms of this type appear only in the first two approximations φ^1, φ^2 to the exact potential. Therefore, the contribution of these terms must be expressed in terms of the average parameters of the fluid. Using (3.4) for this purpose, we find

$$\begin{aligned} \left\langle \frac{\partial (\varphi^1 + \varphi^2)}{\partial x_n} \right\rangle = \langle v_{lx_n} \rangle + \frac{R^3}{2} \int_{r_2} \frac{1}{\frac{4}{3} \pi R^3} \Delta v (\mathbf{r}_2, t) A^2_{2x_n} \alpha_2 (\mathbf{r}_2) (g_l (|\mathbf{r}_2|) - \\ - g(|\mathbf{r}_2|)) \, d^3 r_2 + \frac{R^3}{2} \Delta v (\mathbf{r}_1, t) A^1_{2x_n}. \end{aligned}$$

After rather complicated calculations, we obtain expressions for $\langle \partial(\varphi^1 + \varphi^2)/\partial z$§ and $\langle \partial(\varphi^1 + \varphi^2)/\partial y \rangle = \langle \partial(\varphi^1 + \varphi^2)/\partial x \rangle$.

In order to calculate the contribution to $\langle \partial \varphi / \partial x_n \rangle$ from the higher-order approximations to the exact potential, we must solve numerically the problem of the motion of two bubbles in an infinite fluid. We limit ourselves to the first approximation (after φ^1 and φ^2) to the exact potential. Unlike the case of F_1 (Sec. 3), this approximation gives a zero contribution to F_2. The final formula (with the inclusion of the first correction) has the form

$$\left\langle \frac{\partial \varphi}{\partial z} \right\rangle \Big|_{\Gamma_1} = v_{lz}|_{\mathbf{r}=\mathbf{r}_1} + \frac{\Delta v(\mathbf{r}_1, t)}{2}(3\cos^2\alpha - 1) - \tag{4.4}$$

$$- \frac{3}{2}\frac{\partial(\alpha_2 \Delta v)}{\partial z} R \cos\alpha \sin^2\alpha,$$

$$\left\langle \frac{\partial \varphi}{\partial x} \right\rangle \Big|_{\Gamma_1} = \frac{3}{2}\Delta v(\mathbf{r}_1, t)\cos\alpha\sin\alpha\cos\beta + \frac{3}{2}\frac{\partial(\alpha_2\Delta v)}{\partial z} R\cos\beta\sin\alpha\cos^2\alpha,$$

where R, α, and β are the coordinates of a point on the surface of the bubble at which the velocity is defined in spherical coordinates with the origin at the center of the first bubble and the polar axis directed along Δv.

Using (4.4) we can calculate the contribution to F_2 of the first term of (4.2)

$$\int_{\Gamma_1} \langle \nabla\varphi \rangle^2 \cos\theta d^2 s = \frac{4}{3}\pi R^3 \frac{\partial}{\partial z}(\alpha_2 \Delta v)\left(-\frac{6}{5}\langle v_{lz}\rangle_l + \frac{3}{5}\Delta v\right). \tag{4.5}$$

In order to calculate the contribution to F_2 from the second term of (4.2) we substitute the first approximation to the exact solution of the two-bubble problem into (4.2). After rather lengthy calculations we find

$$\int_{\Gamma_1} \langle (\nabla\varphi')^2 \rangle \cos\theta d^2 s = \frac{\pi R^3}{5}\frac{\partial}{\partial z}(\alpha_2 (\Delta v)^2). \tag{4.6}$$

Finally, from (2.10), (3.16), (4.1), (4.2), (4.5), (4.6), and the equation of continuity we obtain the following expression for the average force acting on a bubble in a dispersed medium with a random distribution of inclusions:

$$\langle F \rangle = \frac{4}{3}\pi R^3 \rho \left[\left(\frac{d_l \langle v_{lz}\rangle_l}{dt} + g \right) - \frac{1}{2}\left(\frac{d_2 \langle v_{2z}\rangle}{dt} - \frac{d_l \langle v_{lz}\rangle_l}{dt} \right) m - \right. \tag{4.7}$$

$$\left. - 0{,}6(\Delta v)^2 \frac{\partial \alpha_2}{\partial z} - 0{,}9\alpha_2 \Delta v \frac{\partial \Delta v}{\partial z} \right],$$

$$\frac{d_l}{dt} = \frac{\partial}{\partial t} + \langle v_{lz}\rangle_l \frac{\partial}{\partial z}, \quad \frac{d_2}{dt} = \frac{\partial}{\partial t} + \langle v_{2z}\rangle \frac{\partial}{\partial z}, \quad m = 1.$$

The first term on the right-hand side of (4.7) is the usual buoyancy force, calculated with respect to the gravitational acceleration of the fluid; the second is the effective mass force acting on the bubble. It is the same as the force acting on a single bubble in an infinite fluid. Recall that in the derivation of (4.7) we used only the first approximation (after φ^1 and φ^2) to the exact solution of the two-bubble problem. The effective mass was calculated exactly in [2] and it was found that $m = 1 + 0.092\alpha_2$. The third and fourth terms in (4.7) give the contributions to the force which are linear in the volume concentration of bubbles. These terms are due to collective interactions between the inclusions and depend on the derivatives of the average quantities. They do not have analogs in the expression for the force acting on a single bubble.

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RELATIONSHIP BETWEEN THE FLOW PARAMETERS OF CONCENTRATED HIGH-POLYMER SOLUTIONS AND THEIR MEAN-STATISTICAL ORIENTATIONAL STRUCTURE

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The questions of the quality of polymer processing, for example by the methods of extrusion technology, and the production of high modular polymeric materials largely depend on the existence of a monitorable relationship between the external parameters of the processing of polymer media and their internal microstructure.

In this work this problem is analyzed for concentrated solutions of high-polymers (CSH) with the help of the structural—phenomenological model of polymers [1, 2]. The model is based on structural representations of the uniform, isotropic fluctuation network (Fig. 1), describing the specific structural features of CSH, consisting of the fact that in the single-relaxation approximation the concentrated polymer solution in a low-molecular solvent is modeled as a collection of statistically distributed effective network sites (segments) of rubbing with the solvent, spatially linked with one another by elastic subchains with kinetic rigidity. Kinetic rigidity refers to the well-known fact that a subchain cannot assume some conformations by a relative displacement of its tips.

The determination of the mean-statistical orientational structures of CSH follows from their rheodynamic description and the construction of the model. An important question addressed by the microscopic description of CSH is taking into account the interaction of the structural units of flow with their environment in the field of shear, entropic, and diffusion forces. In this case, this is the interaction of the randomly distributed network sites of rubbing with the solvent, linked with one another by elastic subchains with rigidity. Unlike theories for weakly concentrated solutions of polymers with solitary macromolecules [3-5] the interaction is sought relative to the center of mass 0 of the sites x_i^{α}, neighboring the x_i chosen for the analysis. The result is the mean stress, i.e., the reaction to external effect G of the statistically distributed rubbing sites interacting with one another (by means of the bonds) and with the solvent.

The physics of the phenomenon is as follows. At rest, the effective site of CSH, driven by Brownian forces of thermal motion and entropy forces pulling toward the center, undergoes around the center of mass a random walk with a rapidly decreasing Gaussian probability density distribution function. The matrix of the components of the moments of the probability density distribution function of such a walk has an equivalent diagonal form, and the radius vector (Fig. 1) of the deviation equals $x = 0.25sb^2$ (b is the length of the segment of rubbing, s is the number of segments in the subchain of the macromolecule).

Under the action of the external ordering forces, aside from these forces a Stokes friction force, owing to the defect in the velocity of the random walk of the site and the solvent at this point, and an internal friction force, associated with the fact that the segments of the subchains cannot assume all possible conformations by means of the relative displacement of their tips (the property of kinetic rigidity), act on the site.

In this case of distinguished directions, the probability density distribution function of the position of the sites will become distorted. The radius vector, the probability of whose length characterizes the deviation of the sites from the position of equilibrium, in

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